A Second Paper on Perpetuants.

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I here continue the investigation of Perpetuants commenced in Vol. VII, No. 1 of the *American Journal of Mathematics.*

The complete system of the simple or binomial syzygies of the sixth degree is there given, the working out of which led up to the discovery that the simplest sextic perpetuant is of weight 31; for that weight there is one exemplar form, viz.:

$$654^2 3^4$$

and five non-exemplar forms, viz.:

$$6^2 5^2 3^3$$
$$6^2 5 3^4 2$$
$$6^2 4 3^5$$
$$6 5^3 4 3^2$$
$$6 5^2 4 3^3 2\,;$$

the way this came about was that representing the quintic perpetuant forms $54^2 3^4$, $5^3 43^2$, $5^2 43^3 2$, hyper-symbolically by $\overbrace{124}$, $\overbrace{312}$, $\overbrace{213}$ respectively, that although the two combinations

$$\overbrace{213} + 2\ \overbrace{124},$$
$$\overbrace{312} - 2\ \overbrace{124},$$

were both expressible as sextic syzygants, the forms $\overbrace{124}$, $\overbrace{312}$, $\overbrace{213}$ were not each separately so expressible.

Thus far the generating function for sextic perpetuants was shown to be

$$\frac{x^{31} + 0.x^{32} + \ldots}{2.3.4.5.6.},$$

wherein as usual a number μ in the denominator denotes for brevity $(1 - x^{\mu})$. It remains to prove that there are no more terms in the numerator and that the form of the generating function is in reality

$$\frac{x^{31}}{2.3.4.5.6};$$

this amounts to showing that of weights superior to 25 there exist no quintic perpetuant forms, which, not being symbolised by such a symbol as $\overbrace{1+\alpha,\ 2+\beta,\ 4+\gamma}$ (α, β, γ being any positive, including zero, integers) are not singly connected through such forms with sextic syzygies; in other words we have to show that every quintic perpetuant not of the above form is expressible by means of such forms as a sextic syzygant; for this will prove that all exemplar sextic perpetuants are comprised in the symbol $\quad 6^* 5^{1+\alpha} 4^{2+\beta} 3^{4+\gamma}$,
and that consequently the numerator of the generating function is in truth monomial. Firstly, consider the syzygy B_7 of Class 1, Group 5, in the paper above referred to; this is

$$B_7 \qquad 43^5 2^{\kappa-7}.2^2 - 3^6 2^{\kappa-7}.32 \equiv (\kappa-6)\,\overbrace{114} + \overbrace{116} + 2\,\overbrace{124} + \overbrace{213},$$

wherein on the dexter side, reducible quintic forms and forms of lower degree are omitted; in Mr. Hammond's notation we have the operator

$$D_\lambda = \frac{1}{\lambda!}\left(\frac{d}{da_1} + a_1\frac{d}{da_2} + a_2\frac{d}{da_3} + \ldots\right)^\lambda$$

and $\qquad D_\lambda(\lambda_1, \mu_1, \nu_1, \pi_1 \ldots)(\lambda_2, \mu_2, \nu_2, \pi_2 \ldots)(\lambda_3, \mu_3, \nu_3, \pi_3 \ldots)\ldots$

$$= \Sigma(\mu_1, \nu_1, \pi_1 \ldots)(\lambda_2, \nu_2, \pi_2 \ldots)(\lambda_3, \mu_3, \pi_3 \ldots)\ldots$$

where $\qquad \lambda_1 + \mu_2 + \nu_3 + \ldots = \lambda$,
the summation including all (including zero) solutions of this equation; so that for instance $\qquad D_4(4^2 3^2 2^3.2^2) = 43^2 2^3.2^2 + 4^2 3^2 2^2.2$;
take then the operator $D_4 D_3^2$ and operate on each side of the syzygy B_7; thus, putting $\kappa + 5$ for κ to keep the weight $= 2\kappa + 9$, we get

$$3^3 2^{\kappa-2}.2^2 + 43^3 2^{\kappa-3}.2 - 3^4 2^{\kappa-3}.3 - 2(3^5 2^{\kappa-3})$$
$$\equiv (\kappa-1)\,\overbrace{102} + \overbrace{104} + 2\,\overbrace{112} + \overbrace{201}.$$

or since $\overbrace{102}$ and $\overbrace{104}$ are reducible forms, this may be written

$$43^3 2^{\kappa-3}.2 - 3^4 2^{\kappa-3}.3 \equiv 2\,\overbrace{112} + \overbrace{201} \tag{1}$$

but the sinister being a sextic syzygy it must be possible to express it in terms of exemplar quintic forms, quintic compounds and forms of lower degree; in fact reference to the tables before referred to shows the syzygy

$$A_5 \qquad 43^3 2^{\kappa-3}.2 - 3^4 2^{\kappa-3}.3 \equiv \overbrace{112};$$

whence combining A_5 with (1) we have

$$\overbrace{112} + \overbrace{201} \text{ reducible,}$$

which is well known from the previous tables which give the reductions of all

the non-exemplar quintic forms by aid of the exemplar; from the syzygy B_7, then, has been derived the formula which gives the reduction of the non-exemplar quintic perpetuant $\overbrace{201}$, and this must necessarily have been so since the syzygy B_7 includes only one form which becomes a non-exemplar quintic perpetuant when operated upon by $D_4 D_3^2$.

Secondly consider the syzygy

$$C_7 \qquad 43^5 2^{\kappa-8}.2^3 - 3^6 2^{\kappa-8}.32^2 \equiv \tfrac{1}{2}(\kappa-6)(\kappa-7)\overbrace{114} + (\kappa-11)\overbrace{116}$$
$$+ 2(\kappa-8)\overbrace{124} + 2\ \overbrace{126} + 3\ \overbrace{134} + (\kappa-7)\overbrace{213} + \overbrace{215} + 2\ \overbrace{223} + \overbrace{312}.$$

Operating with $D_4 D_3^2$ and comparing with B_5

$$2\ \overbrace{211} + \overbrace{300} + (\kappa-2)\overbrace{201} + \overbrace{203} \equiv -2\ \overbrace{122} - \overbrace{114} - (\kappa-3)\overbrace{112};$$

since $\overbrace{201} \equiv -\overbrace{112}$,

and from taking B_9 and A_7 together

$$\overbrace{203} \equiv -\overbrace{114},$$

this reduces to $\qquad 2\ \overbrace{211} + \overbrace{300} \equiv -2\ \overbrace{122} + 3\ \overbrace{112},$

which does not exhibit the reductions of the forms $\overbrace{211}$, $\overbrace{300}$ by aid of exemplars, but only the reduction of the combination

$$2\ \overbrace{211} + \overbrace{300};$$

and moreover it will be found impossible to so exhibit each separately by consideration of the binomial syzygies; but as a matter of fact we know that each is separately so expressible and it follows that there must exist capitation syzygies which, in conjunction with the binomial syzygies, will enable such reduction to be exhibited; that is to say, there must exist a syzygy which involves the form $\overbrace{223}$ and no other form which is convertible into a non-exemplar quintic perpetuant through the operation of the operator $D_4 D_3^2$.

It appears from this argument, which is a general one, that syzygies must exist containing one and only one form which the operator $D_4 D_3^2$ converts into a quintic non-exemplar perpetuant; each such form therefore must be expressible in terms of sextic compounds, quintic perpetuants of the form $\overbrace{1+\alpha,\ 2+\beta,\ 4+\gamma}$, and quintic perpetuants which the operator $D_4 D_3^2$ converts into directly reducible quintic forms; as these latter perpetuants have all been exhibited as sextic syzygants (vide Table of Syzygies, *American Journal of Mathematics*, Vol. VII, No. 1) we have the theorem as follows:

"Each quintic perpetuant of an exemplar form which is convertible to the non-exemplar form by the operation of the operator $D_4 D_3^2$ can, in combination with quintic perpetuants of the form $\overbrace{1+\alpha,\ 2+\beta,\ 4+\gamma,}$ be expressed as a sextic syzygant."

It results therefore by a sextic capitation that every sextic form is reducible by the aid of such forms as $6^* 5^{1+\alpha} 4^{2+\beta} 3^{4+\gamma}$, and that the only exemplar sextic forms are of this type.

Hence their generating function is

$$\frac{x^{31}}{2.3.4.5.6.},$$

and the generating function for sextic syzygies is

$$\frac{x^6 + x^{13} - 2x^{16} - x^{18} + x^{31}}{2.3.4.5.6.}.$$

§2. Proceeding to consider the perpetuants of the seventh degree, or say the septic perpetuants, it is obvious that a form $76^\kappa 5^\lambda 4^\mu 3^\nu$ will be such, provided only that the sextic form $6^\kappa 5^\lambda 4^\mu 3^\nu$ be singly inexpressible as a septic syzygant.

Suppose the whole series of septic syzygies to be written down and the non-exemplars to be expressed in terms of exemplars as they arise; conceive the operation $D_5 D_4^2 D_3^4$ to be performed throughout on each; this will result in a series of identities and syzygies of the sixth and seventh degrees respectively, and the septic syzygies can be reduced by means of the original syzygies to sextic identities, as in the previous case discussed; as before, exemplar and non-exemplar sextic perpetuant forms will occur, and we must be able to exhibit the reduction of each non-exemplar sextic perpetuant form by the aid of the exemplars; not only so but we must be able to obtain the reduction of every reducible sextic form whatever in a similar manner; *ex. gr.* we have the binomial septic syzygy of weight $2\kappa + 31$:

$$\begin{aligned} 54^3 3^4 2^\kappa . 2 - 4^4 3^4 2^\kappa . 3 &= 654^2 3^4 2^\kappa + 2(5^2 4^3 3^3 2^\kappa) + 3(54^4 3^4 2^{\kappa-1}) \\ &+ (\kappa+1)\, 54^3 3^4 2^{\kappa+1} + 54^5 3^2 2^\kappa - 4^5 3^3 2^\kappa . 2 - 5(4^4 3^5 2^\kappa) \\ &+ 6(4^6 3^3 2^{\kappa-1}) + (\kappa+1)\, 4^5 3^3 2^{\kappa+1}. \end{aligned}$$

and operating with $D_5 D_4^2 D_3^4$ and transposing

$$62^\kappa = 42^\kappa . 2 - 2(4^2 2^{\kappa-1}) - (\kappa+1)\, 42^{\kappa+1},$$

giving the reduction of the sextic form 62^κ.

Just then as in the former case there was a one to one correspondence between the reducible quintic forms and the sextic syzygies, of a weight higher by ten, that involved quintic perpetuants, so in this case we have a correspondence

between the reducible sextic forms and the septic syzygies that involve sextic perpetuants of a weight higher by 25; thus the generating function for reducible sextic form's being
$$\frac{x^6 - x^{31}}{2.3.4.5.6},$$
that for septic syzygies involving sextic perpetuants is
$$\frac{x^{31} - x^{56}}{2.3.4.5.6},$$
and therefore the generating function for sextic perpetuants which are not septic syzygants is
$$\frac{x^{31}}{2.3.4.5.6} - \frac{x^{31} - x^{56}}{2.3.4.5.6} = \frac{x^{56}}{2.3.4.5.6};$$
consequently the theory of capitation shows us that the generating function for septic perpetuants is
$$\frac{x^{63}}{2.3.4.5.6.7}$$

The form $765^2 4^4 3^8$ may be taken as the exemplar septic form of weight 63, and then every exemplar septic form of higher weight includes these numbers in its symbol.

The reasoning above employed is perfectly general and leads easily to the conclusion that the generating function for perpetuants of degree n is, $(n > 2)$,
$$\frac{x^{2^{n-1}-1}}{2.3.4 \ldots . n.};$$
because by operating on the n^{ic} syzygies with the D symbol which corresponds to the simplest $(n - 2)^{ic}$ perpetuant which is not an $(n - 1)^{ic}$ syzygant, we can obtain the identities which give the reduction of every $(n - 1)^{ic}$ reducible form.

The simplest exemplar n^{ic} perpetuant, $(n > 2)$, may be taken of the form
$$n . n - 1 . \overline{n - 2}^2 . \overline{n - 3}^4 . \overline{n - 4}^8 \ldots 3^{2^{n-4}}.$$

The complete system of groundforms to the quantic of unlimited order, the degree being θ and the weight w, may be stated as the coefficient of $a^\theta x^w$ in the development in ascending powers of x of
$$a + a^2 \frac{x^2}{2} + a^3 \frac{x^3}{2.3} + a^4 \frac{x^7}{2.3.4} + a^5 \frac{x^{15}}{2.3.4.5}$$
$$+ a^6 \frac{x^{31}}{2.3.4.5.6} + a^7 \frac{x^{63}}{2.3.4.5.6.7} + \ldots$$
$$+ a^\theta \frac{x^{2^{\theta-1}-1}}{2.3.4 \ldots . \theta} + \ldots$$

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